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2012- MA

Test Paper Code: MA

Time: 3 Hours **Maximum Marks: 300**

INSTRUCTIONS

1. This question-cum-answer booklet has **36** pages and has **29** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question Number in the **Answer Table for Objective Questions**, provided on Page **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **-2 (Negative two)** mark.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

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READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER						

Name:

Test Centre:

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

……………………………………………..

Signature of the Candidate

I have verified the information filled by the Candidate above.

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Signature of the Invigilator

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Special Instructions/ Useful Data

$\mathbb{N}$	:	The set of all natural numbers, that is, the set of all positive integers 1, 2, 3, …
$\mathbb{Z}$	:	The set of all integers
$\mathbb{Q}$	:	The set of all rational numbers
$\mathbb{R}$	:	The set of all real numbers
$\{e_1, e_2, \ldots, e_n\}$	:	The standard basis of the real vector space $\mathbb{R}^n$
f', f''	:	First and second derivatives respectively of a real function f
$f_x(a,b)$, $f_y(a,b)$	:	Partial derivatives with respect to x and y respectively of $f:\mathbb{R}^2 \to \mathbb{R}$ at (a,b)
$R \times S$	:	Product ring of rings R, S with component-wise operations of addition and multiplication

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 Let $\{x_n\}$ be the sequence $+\sqrt{1}, -\sqrt{1}, +\sqrt{2}, -\sqrt{2}, +\sqrt{3}, -\sqrt{3}, +\sqrt{4}, -\sqrt{4}, \ldots$.
If

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \text{ for all } n \in \mathbb{N},$$

then the sequence $\{y_n\}$ is

(A) monotonic
(B) NOT bounded
(C) bounded but NOT convergent
(D) convergent

Q.2 The number of distinct real roots of the equation $x^9 + x^7 + x^5 + x^3 + x + 1 = 0$ is

(A) 1 (B) 3 (C) 5 (D) 9

Q.3 If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

then

(A) $f_x(0,0) = 0$ and $f_y(0,0) = 0$
(B) $f_x(0,0) = 1$ and $f_y(0,0) = 0$
(C) $f_x(0,0) = 0$ and $f_y(0,0) = 1$
(D) $f_x(0,0) = 1$ and $f_y(0,0) = 1$

Q.4 The value of $\int\limits_{z=0}^{1} \int\limits_{y=0}^{z} \int\limits_{x=0}^{y} x\, y^2\, z^3 \, dx\, dy\, dz$ is

(A) $\frac{1}{90}$ (B) $\frac{1}{50}$ (C) $\frac{1}{45}$ (D) $\frac{1}{10}$

Q.5 The differential equation $\left(1+x^2y^3+\alpha x^2y^2\right)dx+\left(2+x^3y^2+x^3y\right)dy=0$ is exact if α equals

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) 2 (D) 3

Q.6 An integrating factor for the differential equation $\left(2xy+3x^2y+6y^3\right)dx+\left(x^2+6y^2\right)dy=0$ is

(A) x^3 (B) y^3 (C) e^{3x} (D) e^{3y}

Q.7 For $c>0$, if $a\hat{i}+b\hat{j}+c\hat{k}$ is the unit normal vector at $\left(1,1,\sqrt{2}\right)$ to the cone $z=\sqrt{x^2+y^2}$, then

(A) $a^2+b^2-c^2=0$
(B) $a^2-b^2+c^2=0$
(C) $-a^2+b^2+c^2=0$
(D) $a^2+b^2+c^2=0$

Q.8 Consider the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers. The order of the element $\frac{2}{3}+\mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ is

(A) 2 (B) 3 (C) 5 (D) 6

Q.9 Which one of the following is TRUE ?

(A) The characteristic of the ring $6\mathbb{Z}$ is 6
(B) The ring $6\mathbb{Z}$ has a zero divisor
(C) The characteristic of the ring $\left(\mathbb{Z}/6\mathbb{Z}\right)\times 6\mathbb{Z}$ is zero
(D) The ring $6\mathbb{Z}\times 6\mathbb{Z}$ is an integral domain

Q.10 Let W be a vector space over $\mathbb{R}$ and let $T:\mathbb{R}^6\to W$ be a linear transformation such that $S=\left\{Te_2, Te_4, Te_6\right\}$ spans W. Which one of the following must be TRUE ?

(A) S is a basis of W
(B) $T\left(\mathbb{R}^6\right)\neq W$
(C) $\left\{Te_1, Te_3, Te_5\right\}$ spans W
(D) $\ker\left(T\right)$ contains more than one element

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Q.11 Consider the following subspace of $\mathbb{R}^3$:

$$W = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, 2x + 2y + z = 0,\ 3x + 3y - 2z = 0,\ x + y - 3z = 0 \right\}.$$

The dimension of W is

(A) 0 (B) 1 (C) 2 (D) 3

Q.12 Let P be a 4×4 matrix whose determinant is 10. The determinant of the matrix $-3P$ is

(A) –810 (B) –30 (C) 30 (D) 810

Q.13 If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 3,$ then the series $\sum_{n=0}^{\infty} a_n x^n$

(A) converges absolutely for $x = -2$
(B) converges but not absolutely for $x = -1$
(C) converges but not absolutely for $x = 1$
(D) diverges for $x = -2$

Q.14 If $Y = \left\{ \frac{x}{1+|x|} \,\middle|\, x \in \mathbb{R} \right\},$ then the set of all limit points of Y is

(A) $(-1,1)$ (B) $(-1,1]$ (C) $[0,1]$ (D) $[-1,1]$

Q.15 If C is a smooth curve in $\mathbb{R}^3$ from $(0,0,0)$ to $(2,1,-1)$, then the value of

$$\int_C (2xy + z)\,dx + (z + x^2)\,dy + (x + y)\,dz$$

is

(A) -1 (B) 0 (C) 1 (D) 2

Space for rough work

Space for rough work

Space for rough work

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Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question Number. Do not write anything else on this page.

Question Number	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

Number of Correct Answers		Marks	(+)
Number of Incorrect Answers		Marks	(–)
Total Marks in Questions 1-15			( )

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Q.16 (a) Examine whether the following series is convergent:

$$\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}.$$ (6)

(b) For each $x \in \mathbb{R}$, let $[x]$ denote the greatest integer less than or equal to x. Further, for a fixed $\beta \in (0,1)$, define $a_n = \frac{1}{n}[n\beta] + n^2\beta^n$ for all $n \in \mathbb{N}$. Show that the sequence $\{a_n\}$ converges to β. (9)

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Q.17 (a) Evaluate $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \sqrt{4+t^3}\, dt}{x^2}$. (6)

(b) For $a, b \in \mathbb{R}$ with $a < b$, let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b). Further, assume that the graph of f intersects the straight line segment joining the points $(a, f(a))$ and $(b, f(b))$ at a point $(c, f(c))$ for $a < c < b$. Show that there exists a real number $\xi \in (a,b)$ such that $f''(\xi) = 0$. (9)

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Q.18 (a) Show that the point (0,0) is neither a point of local minimum nor a point of local maximum for the function $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = 3x^4 - 4x^2y + y^2$ for $(x,y) \in \mathbb{R}^2$. (6)

(b) Find all the critical points of the function $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^3 + y^3 - 3x - 12y + 40$ for $(x,y) \in \mathbb{R}^2$. Also, examine whether the function f attains a local maximum or a local minimum at each of these critical points. (9)

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Q.19 (a) Evaluate $\int_{x=0}^{4} \int_{y=\sqrt{4-x}}^{2} e^{y^3}\, dy\, dx.$ (6)

(b) Using multiple integral, find the volume of the solid region in $\mathbb{R}^3$ bounded above by the hemisphere $z = 1 + \sqrt{1 - x^2 - y^2}$ and bounded below by the cone $z = \sqrt{x^2 + y^2}$. (9)

Q.20 Find the area of the portion of the surface $z = x^2 - y^2$ in $\mathbb{R}^3$ which lies inside the solid cylinder $x^2 + y^2 \leq 1$. (15)

Q.21 Let $y(x)$ be the solution of the differential equation $\dfrac{d^2y}{dx^2} - y = 0$ such that $y(0) = 2$ and $y'(0) = 2\alpha$. Find all values of $\alpha \in [0,1)$ such that the infimum of the set $\{y(x) \mid x \in \mathbb{R}\}$ is greater than or equal to 1. (15)

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Q.22 (a) Assume that $y_1(x) = x$ and $y_2(x) = x^3$ are two linearly independent solutions of the homogeneous differential equation $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + 3y = 0$. Using the method of variation of parameters, find a particular solution of the differential equation $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + 3y = x^5$. (6)

(b) Solve the differential equation $\dfrac{dy}{dx} + \dfrac{5y}{6x} = \dfrac{5x^4}{y^5}$ subject to the condition $y(1) = 1$. (9)

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Q.23 (a) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector field in $\mathbb{R}^3$ and let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $\vec{\nabla} \times \left\{ f\left(|\vec{r}|\right)\vec{r} \right\} = \vec{0}$ for $\vec{r} \neq \vec{0}$. (6)

(b) Let W be the region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Let S be the boundary of W. Using Gauss's divergence theorem, evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$, where

$$\vec{F} = \left(x^2 + y^2 - 4\right)\hat{i} + \left(3xy\right)\hat{j} + \left(2xz + z^2\right)\hat{k}$$

and $\hat{n}$ is the outward unit normal vector to S. (9)

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Q.24 (a) Let G be a finite group whose order is not divisible by 3. Show that for every $g \in G,$ there exists an $h \in G$ such that $g = h^3$. (6)

(b) Let A be the group of all rational numbers under addition, B be the group of all non-zero rational numbers under multiplication and C the group of all positive rational numbers under multiplication. Show that no two of the groups A, B and C are isomorphic. (9)

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Q.25 (a) Let I be an ideal of a commutative ring R. Define

$$A = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Show that A is an ideal of R. (6)

(b) Let F be a field. For each $p(x) \in F[x]$ (the polynomial ring in x over F) define $\varphi : F[x] \to F \times F$ by $\varphi(p(x)) = (p(0), p(1))$.

(i) Prove that φ is a ring homomorphism.

(ii) Prove that the quotient ring $F[x]/(x^2 - x)$ is isomorphic to the ring $F \times F$. (9)

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Q.26 (a) Let P, D and A be real square matrices of the same order such that P is invertible, D is diagonal and $D = PAP^{-1}$. If $A^n = 0$ for some $n \in \mathbb{N}$, then show that $A = 0$. (6)

(b) Let $T : V \to W$ be a linear transformation of vector spaces. Prove the following:

(i) If $\{v_1, v_2, \ldots, v_k\}$ spans V, and T is onto, then $\{Tv_1, Tv_2, \ldots, Tv_k\}$ spans W.

(ii) If $\{v_1, v_2, \ldots, v_k\}$ is linearly independent in V, and T is one-one, then $\{Tv_1, Tv_2, \ldots, Tv_k\}$ is linearly independent in W.

(iii) If $\{v_1, v_2, \ldots, v_k\}$ is a basis of V, and T is bijective, then $\{Tv_1, Tv_2, \ldots, Tv_k\}$ is a basis of W. (9)

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Q.27 (a) Let $\{v_1, v_2, v_3\}$ be a basis of a vector space V over $\mathbb{R}$. Let $T: V \to V$ be the linear transformation determined by

$$Tv_1 = v_1,\ Tv_2 = v_2 - v_3 \text{ and } Tv_3 = v_2 + 2v_3.$$

Find the matrix of the transformation T with $\{v_1 + v_2,\ v_1 - v_2,\ v_3\}$ as a basis of both the domain and the co-domain of T. (6)

(b) Let W be a three dimensional vector space over $\mathbb{R}$ and let $S: W \to W$ be a linear transformation. Further, assume that every non-zero vector of W is an eigenvector of S. Prove that there exists an $\alpha \in \mathbb{R}$ such that $S = \alpha I$, where $I: W \to W$ is the identity transformation. (9)

Q.28 (a) Show that the function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=x^2$ for $x\in\mathbb{R}$, is not uniformly continuous. (6)

(b) For each $n\in\mathbb{N}$, let $f_n:\mathbb{R}\to\mathbb{R}$ be a uniformly continuous function. If the sequence $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a function $f:\mathbb{R}\to\mathbb{R}$, then show that f is uniformly continuous. (9)

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Q.29 (a) Let A be a nonempty bounded subset of $\mathbb{R}$. Show that $\{x \in \mathbb{R} \mid x \geq a \text{ for all } a \in A\}$ is a closed subset of $\mathbb{R}$. (6)

(b) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $|x_{n+1} - x_n| < \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$. Show that the sequence $\{x_n\}$ is convergent. (9)

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2012 - MA Objective Part (Question Number 1 – 15)	
Total Marks	**Signature**

Subjective Part					
Question Number	**Marks**		**Question Number**	**Marks**	
16			**23**		
17			**24**		
18			**25**		
19			**26**		
20			**27**		
21			**28**		
22			**29**		
Total Marks in Subjective Part					

Total (Objective Part) :	
Total (Subjective Part) :	
Grand Total :	
Total Marks (in words) :	
Signature of Examiner(s) :	
Signature of Head Examiner(s) :	
Signature of Scrutinizer :	
Signature of Chief Scrutinizer :	
Signature of Coordinating Head Examiner :	

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